# EXACT SOLUTION TO THE SCHRÖDINGER EQUATION FOR THE QUANTUM RIGID BODY

**ZHONG-QI MA**

*Department of College Computer Education, Hu'nan Normal University, Changsha 410081, The People's Republic of China, and Institute of High Energy Physics, Beijing 100039, The People's Republic of China.*

The exact solution to the Schrödinger equation for the rigid body with the given angular momentum and parity is obtained. Since the quantum rigid body can be thought of as the simplest quantum three-body problem where the internal motion is frozen, this calculation method is a good starting point for solving the quantum three-body problems.

Key words: quantum three-body problem, rigid body, Schrödinger equation.

## 1. INTRODUCTION

The three-body problem is a fundamental problem in quantum mechanics, which has not been well solved. The Faddeev equations [1] provide a method for solving exactly the quantum three-body problems. However, only a few analytically solvable examples were found [2]. The accurate direct solution of the three-body Schrödinger equation with the separated center-of-mass motion has been sought based on different numerical methods, such as the finite difference [3], finite element [4], complex coordinate rotation [5], hyperspherical coordinate [6-8], hyperspherical harmonic [9-11] methods, and a large number of works [12-16]. In those numerical methods, three rotational degrees of freedom are not separated completely from the internal ones. In this letter we present a method to separate completely the rotational degrees of freedom and apply it to the quantum rigid body as an example.

The plan of this letter is organized as follows. In Sec. 2 we shall introduce our notations and briefly demonstrate how to separate the rotational degrees of freedom from the internal ones in a quantum three-body problem. The exact solution to the Schrödinger equation for the rigid body with the given angular momentum and parity is obtained in Sec. 3. A short conclusion is given in sec. 4.

## 2. QUANTUM THREE-BODY PROBLEM

Denote by $\mathbf{r_j}$ and by $M_j$, $j = 1, 2, 3$, the position vectors and the masses of three particles in a three-body problem, respectively. The relative masses are $m_j = M_j/M$, where $M$ is the total mass, $M = \sum M_j$. The Laplace operator

in the three-body Schrödinger equation is proportional to $\sum_{j=1}^3 \ m_j^{-1}\triangle_{\mathbf{r}_j}$, where $\triangle_{\mathbf{r}_j}$ is the Laplace operator with respect to the position vector $\mathbf{r}_j$. Introducing the Jacobi coordinate vectors $\mathbf{x}$ and $\mathbf{y}$ in the center-of-mass frame,

$$\mathbf{x} = -\sqrt{\frac{m_1}{m_2+m_3}}\ \mathbf{r_1}, \qquad \mathbf{y} = \sqrt{\frac{m_2 m_3}{m_2+m_3}}\ \left(\mathbf{r_2}-\mathbf{r_3}\right). \tag{1}$$

we obtain the Laplace operator and the total angular momentum operator $\mathbf{L}$ by a direct replacement of variables:

$$\begin{aligned} \triangle = \sum_{j=1}^{3}\ m_j^{-1}\triangle_{\mathbf{r}_j} = \triangle_{\mathbf{x}} + \triangle_{\mathbf{y}}, \\ \mathbf{L} = \sum_{j=1}^{3}\ -i\hbar\mathbf{r}_j \times \bigtriangledown_{\mathbf{r}_j} = \mathbf{L_x} + \mathbf{L_y}, \\ \mathbf{L_x} = -i\hbar\mathbf{x}\times\bigtriangledown_{\mathbf{x}}, \qquad \mathbf{L_y} = -i\hbar\mathbf{y}\times\bigtriangledown_{\mathbf{y}}. \end{aligned} \tag{2}$$

The three-body Schrödinger equation with the separated center-of-mass motion becomes

$$-\left(\hbar^2/2M\right)\{\triangle_{\mathbf{x}}+\triangle_{\mathbf{y}}\}\,\Psi + V\Psi = E\Psi, \tag{3}$$

where $V$ is a pair potential, depending only upon the distance of each pair of particles.

In the hyperspherical harmonic method [11], for example, two Jacobi coordinate vectors are expressed in their spherical coordinate forms,

$$\mathbf{x} \sim (\rho\cos\omega, \theta_x, \varphi_x), \qquad \mathbf{y} \sim (\rho\sin\omega, \theta_y, \varphi_y). \tag{4}$$

where $\rho$ is called the hyperradius and $\Omega(\omega, \theta_x, \varphi_x, \theta_y, \varphi_y)$ are the five hyperangular variables. The wave function is presented as a sum of products of a hyperradial function and the hyperspherical harmonic function,

$$\Psi_{\ell m}(\mathbf{x},\mathbf{y}) = \sum_{K,\ell_x\ell_y} \psi_{K,\ell_x\ell_y}(\rho)\mathcal{Y}^{\ell m}_{K,\ell_x\ell_y}(\Omega).$$

There is huge degeneracy of the hyperspherical basis, and the matrix elements of the potential have to be calculated between different hyperspherical harmonic states [10], because the interaction in the three-body problem is not hyperspherically symmetric.

The quantum rigid body (top) can be thought of as the simplest quantum three-body problem where the internal motion is frozen. To solve exactly the Schrödinger equation for the rigid body is the first step for solving exactly the quantum three-body problems. Wigner first studied the exact solution for the quantum rigid body (see P.214 in [17]) from the group theory. He characterized the position of the rigid body by the three Euler angles $\alpha$, $\beta$, $\gamma$ of the rotation which brings the rigid body from its normal position into the position in question, and obtained the exact solution for the quantum rigid body, which is nothing but the Wigner $D$-function. For the quantum

three-body problems, as in the helium atom, he separated three rotational degrees of freedom from three internal ones (see Eq. (19.18) in [17]):

$$\Psi_{\ell m}(\mathbf{r_1}, \mathbf{r_2}) = \sum_{\nu} D^{\ell}_{m\nu}(\alpha, \beta, \gamma)^* \psi_{\nu}(r_1, r_2, \omega), \tag{5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinate vectors of two electrons, $\omega$ is their angle, and the Wigner $D$-function form [17] has been replaced with the usual $D$-function form [18]. Wigner did not write the three-body Schrödinger equation explicitly. As a matter of fact, the three-body Schrödinger equation (3) becomes very complicated if one replaces two coordinates vectors of electrons with the Euler angles as well as $r_1$, $r_2$, and $\omega$ for the internal motion. On the other hand, Wigner's idea, to separate the degrees of freedom completely from the internal ones, is helpful to simplify the calculation for the quantum three-body problem. Hsiang and Hsiang in their recent paper [19] also presented the similar idea. In this letter we will develop the idea of Wigner and obtain the exact solution of the Schrödinger equation for the rigid body without introducing the Euler angles directly. This calculation method is a good starting point for solving the quantum three-body problems [19,20].

The Schrödinger equation (3) is spherically symmetric so that its solution can be factorized into a product of an eigenfunction of the angular momentum $\mathbf{L}$ and a "radial" function, which only depends upon three variables, invariant in the rotation of the system:

$$\xi_1 = \mathbf{x} \cdot \mathbf{x}, \qquad \xi_2 = \mathbf{y} \cdot \mathbf{y}, \qquad \xi_3 = \mathbf{x} \cdot \mathbf{y}. \tag{6}$$

For the quantum rigid body, the potential makes the internal motion frozen so that those variables $\xi_j$ are constant.

For a particle moving in a central field, the eigenfunction of the angular momentum is the spherical harmonic function $Y^{\ell}_m(\theta, \varphi)$. How to generalize the spherical harmonic function to the three-body problem without introducing the Euler angles directly? As is well known, $\mathcal{Y}^{\ell}_m(\mathbf{x}) = r^{\ell} Y^{\ell}_m(\theta, \varphi)$, where $(r, \theta, \varphi)$ are the spherical coordinates for the position vector $\mathbf{x}$, is a homogeneous polynomial of degree $\ell$ with respect to the components of $\mathbf{x}$, which does not contain $r^2 = \mathbf{x} \cdot \mathbf{x}$ explicitly. $\mathcal{Y}^{\ell}_m(\mathbf{x})$, called the harmonic polynomial in the literature, satisfies the Laplace equation as well as the eigen-equation for the angular momentum.

In the three-body problem there are two Jacobi coordinate vectors $\mathbf{x}$ and $\mathbf{y}$ in the center-of-mass frame. We shall construct the eigenfunctions of the angular momentum as the homogeneous polynomials of degree $\ell$ with respect to the components of $\mathbf{x}$ and $\mathbf{y}$, which do not contain $\xi_j$ explicitly. According to the theory of angular momentum [18], they are

$$\begin{array}{c} \mathcal{Y}^{\ell q}_{Lm}(\mathbf{x}, \mathbf{y}) = \displaystyle\sum_{\mu} \mathcal{Y}^{q}_{\mu}(\mathbf{x}) \mathcal{Y}^{\ell - q}_{m-\mu}(\mathbf{y}) \langle q, \mu, \ell - q, m - \mu | q, \ell - q, L, m \rangle, \\ 0 \leq q \leq \ell, \quad \text{when} \quad L = \ell, \quad \text{and} \quad 1 \leq q \leq \ell - 1, \quad \text{when} \quad L = \ell - 1. \end{array} \tag{7}$$

where $\langle q, \mu, \ell - q, m - \mu | q, \ell - q, L, m \rangle$ are the Clebsch-Gordan coefficients. The remained combinations with the angular momentum $L < \ell - 1$ contain

the factors $\xi_3$ explicitly [20]. In other words, the eigenfunctions of the total angular momentum $\mathbf{L}^2$ with the eigenvalue $\ell(\ell+1)$, not containing the factors $\xi_j$ explicitly, are those homogeneous polynomials of degree $\ell$ or degree $(\ell+1)$. Let us introduce a parameter $\lambda = 0$ or 1 to identify them:

$$\begin{aligned} \mathcal{Y}^{(\ell+\lambda)q}_{\ell m}(\mathbf{x},\mathbf{y}) &= \sum_{\mu} \mathcal{Y}^{q}_{\mu}(\mathbf{x})\mathcal{Y}^{\ell-q+\lambda}_{m-\mu}(\mathbf{y}) \\ &\quad \times \langle q,\mu,\ell-q+\lambda,m-\mu|q,\ell-q+\lambda,\ell,m\rangle, \\ &\quad \lambda = 0 \text{ and } 1, \qquad \lambda \leq q \leq \ell. \end{aligned} \tag{8}$$

$\mathcal{Y}^{(\ell+\lambda)q}_{\ell m}(\mathbf{x},\mathbf{y})$ is the common eigenfunction of $\mathbf{L}^2$, $L_3$, $\mathbf{L}^2_{\mathbf{x}}$, $\mathbf{L}^2_{\mathbf{y}}$, $\triangle_{\mathbf{x}}$, $\triangle_{\mathbf{y}}$, $\triangle_{\mathbf{xy}}$, and the parity with the eigenvalues $\ell(\ell+1)$, $m$, $q(q+1)$, $(\ell-q+\lambda)(\ell-q+\lambda+1)$, 0, 0, 0, and $(-1)^{\ell+\lambda}$, respectively, where $\mathbf{L}^2$ and $L_3$ are the total angular momentum operators, $\mathbf{L}^2_{\mathbf{x}}$ and $\mathbf{L}^2_{\mathbf{y}}$ are the "partial" angular momentum operators [see Eq. (2)], $\triangle_{\mathbf{x}}$ and $\triangle_{\mathbf{y}}$ are the Laplace operators respectively with respect to the Jacobi coordinate vectors $\mathbf{x}$ and $\mathbf{y}$, and $\triangle_{\mathbf{xy}}$ is defined as

$$\triangle_{\mathbf{xy}} = \frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial x_2 \partial y_2} + \frac{\partial^2}{\partial x_3 \partial y_3}. \tag{9}$$

Because of the conservation of the angular momentum and parity, the solution $\Psi_{\ell m\lambda}(\mathbf{x},\mathbf{y})$ of the Schrödinger equation (3) can be expanded in terms of $\mathcal{Y}^{(\ell+\lambda)q}_{\ell m}(\mathbf{x},\mathbf{y})$, where the conserved quantum numbers $\ell$, $m$ and $\lambda$ are fixed. Since those equations are independent of $m$, we can calculate them by setting $m=\ell$, where [18]

$$\begin{aligned} &\mathcal{Y}^{\ell q}_{\ell\ell}(\mathbf{x},\mathbf{y}) \\ &\quad = (-1)^{\ell}\left\{\frac{[(2q+1)!(2\ell-2q+1)!]^{1/2}}{q!(\ell-q)!2^{\ell+2}\pi}\right\}(x_1+ix_2)^q(y_1+iy_2)^{\ell-q}, \\ &\mathcal{Y}^{(\ell+1)q}_{\ell\ell}(\mathbf{x},\mathbf{y}) \\ &\quad = (-1)^{\ell}\left\{\frac{(2q+1)!(2\ell-2q+3)!}{2q(\ell-q+1)(\ell+1)}\right\}^{1/2}\left\{(q-1)!(\ell-q)!2^{\ell+2}\pi\right\}^{-1} \\ &\qquad \times\ (x_1+ix_2)^{q-1}(y_1+iy_2)^{\ell-q}\left\{(x_1+ix_2)y_3 - x_3(y_1+iy_2)\right\}^{\lambda}. \end{aligned} \tag{10}$$

By substituting $\Psi_{\ell\ell\lambda}(\mathbf{x},\mathbf{y})$ into Eq. (3), a system of the partial differential equations for the coefficients can be obtained. The partial differential equations will be simplified if one changes the normalization factor of $\mathcal{Y}^{(\ell+\lambda)q}_{\ell\ell}(\mathbf{x},\mathbf{y})$, namely $\mathcal{Y}^{(\ell+\lambda)q}_{\ell\ell}(\mathbf{x},\mathbf{y})$ in Eq. (11) is replaced by $Q^{\ell\lambda}_q(\mathbf{x},\mathbf{y})$, which is proportional to $\mathcal{Y}^{(\ell+\lambda)q}_{\ell\ell}(\mathbf{x},\mathbf{y})$:

$$\begin{aligned} \Psi_{\ell\ell\lambda}(\mathbf{x},\mathbf{y}) &= \sum_{q=\lambda}^{\ell} \psi^{\ell\lambda}_q(\xi_1,\xi_2,\xi_3)Q^{\ell\lambda}_q(\mathbf{x},\mathbf{y}), \\ Q^{\ell\lambda}_q(\mathbf{x},\mathbf{y}) &= \{(q-\lambda)!(\ell-q)!\}^{-1}(x_1+ix_2)^{q-\lambda}(y_1+iy_2)^{\ell-q} \\ &\quad \times\ \{(x_1+ix_2)y_3 - x_3(y_1+iy_2)\}^{\lambda} \\ &\quad \lambda = 0,1, \qquad \lambda \leq q \leq \ell. \end{aligned} \tag{11}$$

The partial differential equations for the functions $\psi_q^{\ell\lambda}(\xi_1,\xi_2,\xi_3)$ are:

$$
\begin{aligned}
&-\frac{\hbar^2}{2M}\left\{\triangle\psi_q^{\ell\lambda}+4q\frac{\partial\psi_q^{\ell\lambda}}{\partial\xi_1}+4(\ell-q+\lambda)\frac{\partial\psi_q^{\ell\lambda}}{\partial\xi_2}+2(q-\lambda)\frac{\partial\psi_{q-1}^{\ell\lambda}}{\partial\xi_3}\right.\\
&\qquad\left.+2(\ell-q)\frac{\partial\psi_{q+1}^{\ell\lambda}}{\partial\xi_3}\right\}=(E-V)\psi_q^{\ell\lambda},\\
&\qquad\lambda\leq q\leq\ell,\qquad\lambda=0,1.
\end{aligned}
\tag{12}
$$

This system of the partial differential equations was first obtained by Hsiang and Hsiang [19]. It is a good starting point for solving the quantum three-body problems [19,20].

## 3. QUANTUM RIGID BODY

For the quantum rigid body, the potential preserves the geometrical form of the rigid body fixed. It can be replaced by the constraints:

$$
\xi_1=\text{const.}\qquad\xi_2=\text{const.}\qquad\xi_3=\text{const.}\tag{13}
$$

Therefore, the solution of the Schrödinger equation for the quantum rigid body can be expressed as

$$
\Psi_{\ell\ell\lambda}(\mathbf{x},\mathbf{y})=\sum_{q=\lambda}^{\ell}f_q^{\ell\lambda}Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y}).\tag{14}
$$

where $f_q^{\ell\lambda}$ are constant. Recall that $Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y})$ is the solution of the Laplace equation. Due to the constraints (13) some differential terms with respect to $\xi_j$ in the Laplace equation should be removed so that the Laplace equation is violated, namely, the rigid body obtains an energy $E$. On the other hand, as a technique of calculation, we can calculate those differential terms first where $\xi_j$ are not constant, and then set the constraints (13). The contribution from those terms is nothing but the minus energy $-E$ of the rigid body.

In the calculation, we first separate the six Jacobi coordinates [see Eq. (4)] into three rotational coordinates and three internal coordinates. The lengths of $\mathbf{x}$ and $\mathbf{y}$ and their angle $\omega$ are

$$
r_x=\sqrt{\xi_1},\qquad r_y=\sqrt{\xi_2},\qquad\cos\omega=\xi_3/\sqrt{\xi_1\xi_2}.\tag{15}
$$

Obviously, those three variables are also constant in the constraints (13). Assume that in the normal position of the rigid body the Jacobi coordinate vector $\mathbf{x}$ is along the $Z$ axis and $\mathbf{y}$ is located in the $XZ$ plane with a positive $X$ component. A rotation $R(\alpha,\beta,\gamma)$ brings the rigid body from its normal position into the position in question. The Euler angles $\alpha$, $\beta$, and $\gamma$ describe the rotation of the rigid body. The definition for the Euler angles are different from that of Wigner (see Eq. (7) and Ref. [17]) because $\mathbf{x}$ and $\mathbf{y}$ here are

the Jacobi coordinate vectors. To shorten the notations, we define

$$\begin{array}{lll} c_\alpha = \cos\alpha, & c_\beta = \cos\beta, & c_\gamma = \cos\gamma, \\ c_x = \cos\theta_x, & c_y = \cos\theta_y, & C = \cos\omega, \\ s_\alpha = \sin\alpha, & s_\beta = \sin\beta, & s_\gamma = \sin\gamma, \\ s_x = \sin\theta_x, & s_y = \sin\theta_y, & S = \sin\omega. \end{array} \tag{16}$$

According to the definition, we have [18]

$$\begin{array}{l} R(\alpha,\beta,\gamma) = \begin{pmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{pmatrix}, \\ \begin{array}{ll} x_1 + ix_2 = r_x e^{i\alpha} s_\beta, & y_1 + iy_2 = r_y e^{i\alpha}\left(c_\beta c_\gamma S + s_\beta C + i s_\gamma S\right), \\ x_3 = r_x c_\beta, & y_3 = r_y\left(-s_\beta c_\gamma S + c_\beta C\right). \end{array} \end{array} \tag{17}$$

Through the replacement of variables:

$$\begin{array}{l} (r_x, \theta_x, \varphi_x, r_y, \theta_y, \varphi_y) \quad \longrightarrow \quad (r_x, r_y, \omega, \alpha, \beta, \gamma), \\ \alpha = \varphi_x, \qquad\qquad \beta = \theta_x, \\ C = c_x c_y + s_x s_y \cos(\varphi_x - \varphi_y), \\ \cot\gamma = \dfrac{s_x c_y - c_x s_y \cos(\varphi_x - \varphi_y)}{s_y \sin(\varphi_x - \varphi_y)}, \end{array} \tag{18}$$

we obtain

$$\begin{array}{rcl} \triangle_{\mathbf{x}} & = & \dfrac{1}{r_x}\dfrac{\partial^2}{\partial r_x^2} r_x + \cdots, \\ \triangle_{\mathbf{y}} & = & \dfrac{1}{r_y}\dfrac{\partial^2}{\partial r_y^2} r_y + \dfrac{1}{r_y^2 S}\dfrac{\partial}{\partial\omega} S \dfrac{\partial}{\partial\omega} + \cdots, \end{array} \tag{19}$$

where the neglected terms are those differential terms only with respect to the rotational variables $\alpha$, $\beta$ and $\gamma$. Now,

$$\begin{array}{l} \dfrac{\hbar^2}{2M}\left\{\dfrac{1}{r_x}\dfrac{\partial^2}{\partial r_x^2} r_x + \dfrac{1}{r_y}\dfrac{\partial^2}{\partial r_y^2} r_y + \dfrac{1}{r_y^2 S}\dfrac{\partial}{\partial\omega} S\dfrac{\partial}{\partial\omega}\right\}\Psi_{\ell\ell\lambda}(\mathbf{x},\mathbf{y})\Big|_{\xi_j = const.} \\ \quad = E\Psi_{\ell\ell\lambda}(\mathbf{x},\mathbf{y})\big|_{\xi_j = const.}, \end{array} \tag{20}$$

where $\Psi_{\ell\ell\lambda}(\mathbf{x},\mathbf{y})$ is given in Eq. (14).

Through a direct calculation, we obtain

$$\begin{array}{l} \dfrac{1}{r_x}\dfrac{\partial^2}{\partial r_x^2} r_x Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y}) = \dfrac{q(q+1)}{r_x^2} Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y}), \\ \dfrac{1}{r_y}\dfrac{\partial^2}{\partial r_y^2} r_y Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y}) = \dfrac{(\ell - q + \lambda)(\ell - q + \lambda + 1)}{r_y^2} Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y}), \\ \dfrac{1}{r_y^2 S}\dfrac{\partial}{\partial\omega} S \dfrac{\partial}{\partial\omega} Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y}) \\ \quad = \left\{(\ell - q)\left[(\ell - q + 2\lambda)\cot^2\omega - 1\right]\right. \\ \qquad \left. + \lambda\left(\cot^2\omega - 1\right)\right\} Q_q^{\ell\lambda}(\mathbf{x},\mathbf{y})/r_y^2 \\ \qquad - (q - \lambda + 1)\left(2\ell - 2q + 2\lambda - 1\right)\left(C/S^2\right) Q_{q+1}^{\ell\lambda}(\mathbf{x},\mathbf{y})/\left(r_x r_y\right) \\ \qquad + (q - \lambda + 1)(q - \lambda + 2) S^{-2} Q_{q+2}^{\ell\lambda}(\mathbf{x},\mathbf{y})/r_x^2. \end{array} \tag{21}$$

Therefore, the coefficients $f_q^{\ell\lambda}$ satisfies a system of linear algebraic equations with the equation number $(\ell - \lambda + 1)$:

$$
\begin{aligned}
(2ME/\hbar^2)f_q^{\ell\lambda} = & \Big\{q(q+1)/r_x^2 + (\ell-q+\lambda)(\ell-q+\lambda+1)/r_y^2 \\
& + \left[(\ell-q)(\ell-q+2\lambda)\cot^2\omega - (\ell-q) + \lambda\left(\cot^2\omega - 1\right)\right]/r_y^2\Big\} f_q^{\ell\lambda} \\
& - \left\{(q-\lambda)\left(2\ell-2q+2\lambda+1\right)C/\left(S^2 r_x r_y\right)\right\} f_{q-1}^{\ell\lambda} \\
& + \left\{(q-\lambda)(q-\lambda-1)/\left(S^2 r_x^2\right)\right\} f_{q-2}^{\ell\lambda}.
\end{aligned}
\tag{22}
$$

where $r_x$, $r_y$ and $\omega$ are constant.

Due to the spherical symmetry, the energy level with the given total angular momentum $\ell$ is $(2\ell+1)$-degeneracy (normal degeneracy). Furthermore, since $\lambda \leq q \leq \ell$, there are $(\ell+1)$ sets of solutions with the parity $(-1)^\ell$ and $\ell$ sets of solutions with the parity $(-1)^{\ell+1}$. This conclusion coincides with that by Wigner (see P. 218 in [17]). When $\ell = 0$ we have the constant solution with zero energy and even parity. When $\ell = 1$, we have one set of solutions $\Psi_{\ell m 1}$ with the even parity and two sets of solutions $\Psi_{\ell m 0}$ with the odd parity:

$$
\begin{aligned}
& \Psi_{111}(\mathbf{x},\mathbf{y}) = (x_1 + ix_2)y_3 - x_3(y_1 + iy_2), \\
& E_{11} = \hbar^2/\left(Mr_x^2\right) + \hbar^2/\left(2Mr_y^2\sin^2\omega\right), \\
& \Psi_{110}^{(1)} = x_1 + ix_2, \qquad E_{10}^{(1)} = \hbar^2/\left(Mr_x^2\right), \\
& \Psi_{110}^{(2)} = \frac{C}{S^2 r_x r_y}(x_1 + ix_2) + \left(\frac{2}{r_x^2} - \frac{1}{S^2 r_y^2}\right)(y_1 + iy_2), \\
& E_{10}^{(2)} = \hbar^2/\left(2Mr_y^2\sin^2\omega\right),
\end{aligned}
\tag{23}
$$

It is similar to obtain the solutions with the higher orbital angular momentum $\ell$. The partners of the solutions with the smaller eigenvalues of $L_3$ can be calculated from them by the lowering operator $L_-$.

## 4. CONCLUSION

In summary, we have reduced the three-body Schrödinger equation for any given total orbital angular momentum and parity to a system (12) of the coupled partial differential equations with respect only to three variables, describing the internal degrees of freedom in a three-body problem. This equation system is a good starting point for solving the quantum three-body problems. As an example, we obtain the exact solution to the Schrödinger equation for the rigid body.

**Acknowledgements**. The author would like to thank Prof. Hua-Tung Nieh and Prof. Wu-Yi Hsiang for drawing his attention to the quantum three-body problems. This work was supported by the National Natural Science Foundation of China and Grant No. LWTZ-1298 of the Chinese Academy of Sciences.

**REFERENCES**


1. L. D. Faddeev, *Sov. Phys. JETP* **12**, 1014 (1961); *Sov. Phys. Dokl.* **6**, 384 (1961); *Sov. Phys. Dokl.* **7**, 600 (1963).
2. N. Barnea and V. Mandelzweig, *Phys. Rev.* C **49**, 2910 (1994).
3. I. L. Hawk and D. L. Hardcastle, *Comp. Phys. Commun.* **16**, 159 (1979).
4. F. S. Levin and J. Shertzer, *Phys. Rev.* A **32**, 3285 (1985).
5. Y. K. Ho, *Phys. Rev.* A **34**, 4402 (1986).
6. I. G. Fray and B. J. Howard, *Chem. Phys.* **111**, 33 (1987).
7. J. Z. Tang, S. Watanabe, and M. Matsuzawa, *Phys. Rev.* A **46**, 2437 (1992).
8. B. Zhou, C. D. Lin, J. Z. Tang, S. Watanabe, and M. Matsuzawa, *J. Phys.* B **26**, 2555 (1993); B. Zhou and C. D. Lin, *J. Phys.* B **26**, 2575 (1993).
9. M. I. Haftel and V. B. Mandelzweig, *Phys. Lett.* A **120**, 232 (1987).
10. M. I. Haftel and V. B. Mandelzweig, *Ann. Phys.* (N.Y.) **189**, 29 (1989).
11. R. Krivec and V. B. Mandelzweig, *Phys. Rev.* A **42**, 3779 (1990).
12. F. M. Lev, *Fortschritte der Physik*, **31**, 75 (1983).
13. H. Letz, *Nuovo Cimento* B **26**, 522 (1975).
14. E. F. Redish, "Lectures in the Quantum Three-Body Problem", Preprint MDDP-TR-77-060, 1977.
15. J. Ginibre and M. Moulin, "Hilbert Space Approach to the Quantum Mechanical Three-Body Problem", Preprint LPTHE-TH 74/8, 1974.
16. R. Krivec, *Few-Body Systems*, **25**, 199 (1998) and references therein.
17. E. P. Wigner, "Group Theory and its Application to the Quantum Mechanics of Atomic Spectra" (Academic Press, New York 1959).
18. A. R. Edmonds, "Angular Momentum in Quantum Mechanics" (Princeton University Press, 1957).
19. W. T. Hsiang and W. Y. Hsiang, "On the reduction of the Schrödinger's equation of three-body problem to a system of linear algebraic equations", Preprint, 1998.
20. Zhong-Qi Ma and An-Ying Dai, "Quantum three-body problem", Preprint, physics/9905051, 1999.